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HOMOGENEOUSLY TRACEABLE RESULTS IN CLAW-FREE GRAPHS

by

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ABSTRACT

A graph G is *homogeneously traceable* if for each $v \in V(G)$ there is a Hamilton path starting at v. In this paper we find a sufficient condition for a claw-free graph to be homogeneously traceable in terms of a neighbourhood union condition.

Preliminaries

A graph G is said to be *homogeneously traceable* if for each $v \in V(G)$ there is a Hamilton path starting at v. We will call a path a *v-path* if it starts at v.

Theorem 1[3]
If G is a 3-connected, claw-free graph such that

$$|N(u) \cup N(v)| > (2p-5)/3$$

for all nonadjacent pairs of vertices u, v then G is homogeneously traceable. □

Clearly, any graph that is Hamiltonian is also homogeneously traceable.

Theorem 2[4]
If G is a 3-connected, claw-free graph such that

$$|N(u) \cup N(v)| > 11(p-7)/21$$

for all nonadjacent pairs u, v then G is Hamiltonian. □

So Theorem 1 is a corollary of Theorem 2.

Theorem 3[1]
Let G be a 2-connected graph with

$$|N(u) \cup N(v)| \geq p/2$$

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for all nonadjacent pairs of vertices u, v. Then either G is Hamiltonian, or G is equal to the Petersen graph, or G is a spanning subgraph of one of the following families:
a) $K_2 + (K_q \cup K_r \cup K_s)$;
b) $K_1 + (K_q \cup K_r \cup K_s \cup T)$, where $q, r, s \geq 2$ and T is the edge set of a triangle containing exactly one vertex or K_p, K_q and K_r;
c) $K_q \cup K_r \cup K_s \cup T_1 \cup T_2$, where $q, r, s \geq 3$ and T_1 and T_2 are the edge sets of two vertex-disjoint triangles each containing exactly one vertex from K_q, K_r and K_s. □

This Theorem generalises each of Theorems 1 and 2, since none of the exceptional graphs are 3-connected.

In [5] Lindquester investigated the effect of distance on neighbourhood union conditions.

Theorem 4[4]

Let G be a 2-connected graph with

$$|N(u) \cup N(v)| \geq (2p-1)/3$$

for all pairs of vertices u, v at distance 2. Then G is Hamiltonian.

Results

We will obtain a sufficient condition for a 2-connected, claw-free graph to be homogeneously traceable in terms of the neighbourhood union of vertices at distance 2. First, we will need the following Lemma.

Lemma 5

Let G be a 2-connected graph. Let $P = v_1, v_2, ..., v_m$ be a longest v_m path. Then there is a path $P' = u_1, u_2, ..., u_m = v_m$ with $V(P') = V(P)$ such that in P', u_1 is adjacent to some vertex u_{t+1} and not to u_t.

Proof

Let P be a longest v_m-path and suppose that there is no path P' with the required property. Let $Q = u_1, u_2, ..., u_m = v_m$ be a v_m-path with $V(Q) = V(P)$ and the degree of (u_1) as large as possible. Then Q is a longest v_m-path.

Traversing Q from u_1 towards u_m let v_{r+1} be the first vertex to which u_1 is not adjacent. Then u_1 is adjacent to $u_2, u_3, ..., u_r$ and the degree of u_1 is $r-1$. Then u_1 is not adjacent to any other vertices of P else we can put $Q = P'$ and we're done. Since G is 2-connected, u_r cannot be a cut point. Now if one of $u_2, u_3, ..., u_{r-1}$, say u_k, is adjacent to some $y \notin Q$ we will immediately get the longer v_m-path

$$u_m, u_{m-1}, ..., u_{k+1}, u_1, u_2, ..., u_k, y.$$

Thus one of $u_2, u_3, ..., u_{r-1}$, say u_n, is adjacent to a vertex u_q with $q > r$. Note that u_1 is adjacent to u_{n+1}. Take the path

$$W = u_m, u_{m-1}, ..., u_q, u_{q-1}, ..., u_{n+1}, u_1, u_2, ..., u_n.$$

This is also a longest v_m-path with $V(W) = V(P)$. Now if u_n is not adjacent to all of $u_{n-1}, u_{n-2}, \ldots, u_1, u_{n+1}, u_{n+2}, \ldots, u_q$ then we have a path with the required property. On the other hand, if u_n is adjacent to all of these, then the degree of u_n is at least $q-1 > r-1$ and we have a longest v_m-path where the degree of the first vertex, u_n is greater than the degree of the first vertex of Q, contradicting the choice of Q. □

Theorem 6

Let G be a 2-connected, claw-free graph with

$$|N(u) \cup N(v)| > (p-3)/2$$

for all pairs of vertices u, v at distance 2. Then G is homogeneously traceable.

Proof

Let G be a 2-connected, claw-free graph with $|N(u) \cup N(v)| > (p-3)/2$ for every pair of vertices u, v at distance 2. Let $z \in V(G)$. We aim to find a Hamilton path with end vertex z. Let $P = v_1, v_2, \ldots, v_m$, $v_m = z$, be a longest path in G with end vertex z. If $m = p$ we are done, so suppose $m < p$. Then there is a vertex x, $x \notin P$. Since G is 2-connected, there are at least two openly disjoint paths from x to P. Let the two end vertices of any set of such paths with the lowest subscripts be v_k, v_l, where $k < l$. Without loss of generality we can assume $xv_l \in E(G)$. Since G is claw-free and $1 < k < m$, we have $v_{k-1}v_{k+1} \in E(G)$. Now $l \neq k+2$ since if $l = k+2$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, v_{k-1}, v_{k-2}, \ldots, v_1.$$

Thus $l > k+2$.

Now by Lemma 5 we can assume that there is a vertex v_t so that v_1 is adjacent to v_{t+1} and not to v_t. Choose the smallest t for which this happens.

Now $t \neq k$ since this would imply v_1 is adjacent to v_{k+1} and we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_1, v_2, \ldots, v_k, x.$$

Also $t \neq k+1$ since this would imply v_1 is adjacent to v_{k+2} and we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+2}, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, v_k, x.$$

Thus $t \neq k, k+1$.

Traversing P from v_1 towards v_m, let v_{r+1} be the first vertex to which v_1 is not adjacent. Then v_1 is adjacent to $v_2, v_3, \ldots, v_r$, and not adjacent to $v_{r+1}, v_{r+2}, \ldots, v_t$. Now $r < k$ since if v_1 is adjacent to v_k we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_{l-1}.$$

Thus $r < k$.

We will arrive at a contradiction by showing that there is a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{l-1}) \cup N(x))$. Note that v_1, v_t are vertices at distance 2 by the definition of t. Also x, v_{l-1} are distance 2 apart since $xv_l \in E(G)$ and $xv_{l-1} \notin E(G)$.

A-1

Let $y \in N(v_1) \cup N(v_t)$. Suppose $y \notin P$. Then since P is a longest v_m-path, $y \notin N(v_1)$ and if $y \in N(v_t)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, y.$$

Thus we have $y \in P$, and so $y = v_s$ for some s.

We will now consider 2 cases for l:

Case 1: Suppose $l < m$. Note that since G is claw-free and P is a longest v_m-path we have $v_{l-1}v_{l+1} \in E(G)$.

We have already shown that $t \neq k, k+1$. By similar arguments, $t \neq l, l+1$. We will now show $t \neq l-1$. Suppose $t = l-1$. Then v_1 is adjacent to v_l and we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{l+1}, v_{l-1}, v_{l-2}, \ldots, v_1, v_l, x.$$

Thus $t \neq l-1$.

We have also previously shown that $l \neq k+2$. We will now show that $l-1 \neq k+2$. Suppose $l-1 = k+2$. Then we will get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{l+1}, v_{l-1}, v_l, x, v_k, v_{k+1}, v_{k-1}, v_{k-2}, \ldots, v_1.$$

Thus $l-1 \neq k+2$.

Let $v_s \in N(v_1) \cup N(v_t)$. Now clearly $s \neq 1, t$ by the definition of t. We claim $s \neq k, l-1, l$.

Suppose first $s = k$. Now if $v_k \in N(v_1)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_{k-1}, v_{k-2}, \ldots, v_1, x.$$

So suppose $v_k \in N(v_t)$. Then for $t < k$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_{k-1}, v_{k-2}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, v_k, x$$

and for $t > k$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_t, v_k, x.$$

Thus $s \neq k$. By similar arguments, we can show $s \neq l$.

Now suppose $s = l-1$. If $v_{l-1} \in N(v_1)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_1, v_{l-1}, v_{l-2}, \ldots, v_{k+1}.$$

So suppose $v_{l-1} \in N(v_t)$. Then for $t < k$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, v_{l-1}, v_{l-2}, \ldots, v_{k+1},$$

for $k < t < l$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_t, v_{t-1}, \ldots, v_{k+1}$$

and for $t > l$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{l-1}, v_t, v_{t-1}, \ldots, v_l, x.$$

Thus $s \neq l - 1$.

Let $v_s \in N(v_1)$. Now $s \neq k+1$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_1, v_2, \ldots, v_k, x.$$

Similarly, $s \neq l+1$. Now if $v_{s-1} \in N(x)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_1, v_2, \ldots, v_{s-1}, x.$$

Also, $v_{s-1} \notin N(v_{l-1})$ for if it were, for if $1 < s < k$ we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_1, v_s, v_{s+1}, \ldots, v_{k-1},$$

if $k+1 < s < l-1$ we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_1, v_s, v_{s+1}, \ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_{k+1}$$

and if $s > l+1$ we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_1, v_2, \ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_l, x.$$

Thus if $v_s \in N(v_1)$ we have $v_{s-1} \notin N(v_{l-1}) \cup N(x)$.

Again consider $v_s \in N(v_1)$. For $k+1 < s < m$ we have $v_{s+1} \notin N(x)$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{s+1}, x, v_k, v_{k-1}, \ldots, v_1, v_s, v_{s-1}, \ldots, v_{k+1}.$$

For $l+1 < s < m$, $v_{s+1} \notin N(v_{l-1})$ else we get the longer v_m- path

$$v_m, v_{m-1}, \ldots, v_{s+1}, v_{l-1}, v_{l-2}, \ldots, v_1, v_s, v_{s-1}, \ldots, v_l, x.$$

There are now 3 possible locations for t: $1 < t < k$, $k+1 < t < l-1$ and $t > l+1$. We will now consider these 3 cases for t.

Case 1.1: Suppose $t > l+1$.

Let $v_s \in N(v_1) \cup N(v_t)$. We have already shown that $s \neq 1, t, k, l-1, l$. We now claim $s \neq k+1$. Note that we have already shown that $v_{k+1} \notin N(v_1)$. So suppose $v_{k+1} \in N(v_t)$. Then we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_k, x, v_l, v_{l+1}, \ldots, v_t, v_{k+1}, v_{k+2}, \ldots, v_{l-1}.$$

So $v_{k+1} \notin N(v_t)$ and therefore $s \neq k+1$.

We consider 2 subcases:

Case 1.1.1: Assume v_{l-1} is adjacent to some vertex v_q with $q < k$.

First we claim $q \neq k-1$ since if $v_{k-1} \in N(v_{l-1}$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_{k-1}, v_{k-2}, \ldots, v_1.$$

Thus $q < k-1$. Also, $q \neq k-2$ for if $v_{k-2} \in N(v_{l-1})$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_{l-1}, v_{k-2}, v_{k-3}, \ldots, v_1.$$

So we have $q < k-2$.

Recall $v_s \in N(v_1) \cup N(v_t)$. Now $s \neq k-1$ since if $v_{k-1} \in N(v_1)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_q, v_{q+1}, \ldots, v_{k-1}, v_1, v_2, \ldots, v_{q-1}$$

and if $v_{k-1} \in N(v_t)$ then we will get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-1}, v_t, v_{t-1}, \ldots, v_k, x.$$

So $s \neq k-1$.

We will now construct the 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{l-1}) \cup N(x))$.

First suppose $s < k-1$ or $k+1 < s < l-1$. Now $v_{s-1} \notin N(x)$ by the choice of k, l.

Again recall $v_s \in N(v_1) \cup N(v_t)$. Suppose $v_s \in N(v_1)$. We have shown above that $v_{s-1} \notin N(v_{l-1}) \cup N(x)$. Now suppose $v_s \in N(v_t)$. Then $v_{s-1} \notin N(v_{l-1})$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{s-1}. v_{l-1}, v_{l-2}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_l, x.$$

So for the case $s < k$, $k+2 < s < l-1$ let v_{s-1} be the vertex corresponding to v_s in the 1:1 mapping.

Now suppose $l < s < t$ or $t < s < m$.

Suppose $v_s \in N(v_1)$. Then we have shown above that $v_{s+1} \notin N(v_{l-1}) \cup N(x)$. So suppose $v_s \in N(v_t)$. Then $v_{s+1} \notin N(v_{l-1})$ else for $s < t$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{l-1}, v_{s+1}, v_{s+2}, \ldots, v_t, v_s, v_{s-1}, \ldots, v_l, x$$

and for $s > t$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{s+1}, v_{l-1}, v_{l-2}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_l, x.$$

Also $v_{s+1} \notin N(x)$ for if it were for $s < t$ we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_s, v_t, v_{t-1}, \ldots, v_{s+1}, x$$

and for $s > t$ we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{s+1}, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_{k+1}.$$

For the case $l < s < t$, $t < s < m$ let v_{s+1} correspond to v_s in the desired 1:1 mapping.

Note that we have not found an image point corresponding to v_m. We claim to have found a 1:1 mapping from $N(v_1) \cup N(v_t) - v_m$ to $V(G) - (N(v_{l-1}) \cup N(x)) - \{v_{k-2}, v_{k-1}, v_{l-1}, x\}$.

Clearly we have shown a 1:1 mapping from $N(v_1) \cup N(x) - v_m$ to a subset S of $V(G) - (N(v_{l-1}) \cup N(x))$.

We now claim $v_{k-2}, v_{k-1}, v_{l-1}, x \notin S$.

First suppose $v_{k-2} \in S$. Then $v_{k-2} = v_{s-1}$ or v_{s+1} for some s. But $v_{k-2} = v_{s-1}$ implies $s = k-1$ and $v_{k-2} = v_{s+1}$ implies $s = k-3 > l$, both contradictions. Thus $v_{k-2} \notin S$.

Next suppose $v_{k-1} \in S$. Then by the way we have constructed our 1:1 mapping we have $v_{k-1} = v_{s-1}$ or v_{s+1} where $v_s \in N(v_1) \cup N(v_t)$. But $v_{k-1} = v_{s-1}$ implies $s = k$, but we have already shown that $s \neq k$. Also $v_{k-1} = v_{s+1}$ implies $s = k-2 > l$, a contradiction. So $v_{k-1} \notin S$.

Now suppose $v_{l-1} \in S$. Then $v_{l-1} = v_{s-1}$ or v_{s+1} for some s with $v_s \in N(v_1) \cup N(v_t)$. But $v_{l-1} = v_{s-1}$ implies $s = l$, a contradiction and $v_{l-1} = v_{s+1}$ implies $s = l-2 > l$, again giving a contradiction. Hence $v_{l-1} \notin S$.

Finally suppose $x \in S$. Then x is the image point of some y where $y \in N(v_1) \cup N(v_t) - v_m$. But all the image points are on P and $x \notin P$. Thus $x \notin S$.

Now it can be easily seen that $x, v_{l-1} \notin N(v_{l-1}) \cup N(x)$. Also, clearly $v_{k-1} \notin N(x)$ and if $v_{k-1} \in N(v_{l-1})$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_{k-1}, v_{k-2}, \ldots, v_1.$$

Again, by the choice of k, $v_{k-2} \notin N(x)$ and if $v_{k-2} \in N(v_{l-1})$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_{l-1}, v_{k-2}, v_{k-3}, \ldots, v_1.$$

Thus $v_{k-2}, v_{k-1}, v_{l-1}, x \notin N(v_{l-1}) \cup N(x)$.

We get

$$\begin{aligned}(p-3)/2 - 1 &< |N(v_1) \cup N(v_t) - v_m| \\ &\leq |V(G) - (N(v_{l-1}) \cup N(x)) - \{x, v_{k-2}, v_{k-1}, v_{l-1}\}| \\ &< p - (p-3)/2 - 4 = (p-3)/2 - 1\end{aligned}$$

a contradiction.

Case 1.1.2: So we can assume v_{l-1} is not adjacent to any vertex v_q where $q < k$.

Recall that $v_s \in N(v_1) \cup N(v_t)$ and $s \neq 1, t, k, k+1, l-1, l$ We will construct a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{l-1}) \cup N(x))$.

First suppose $s < k$. Then by the choice of k, $v_s \notin N(x)$ and by the hypothesis of Case 1.1.2, $v_s \notin N(v_{l-1})$. So let v_s correspond to itself when $s < k$.

Next suppose $k+1 < s < l-1$. Then by the choice of t, $v_s \notin N(v_1)$ and so assume $v_s \in N(v_t)$. Now $v_{s-1} \notin N(x)$ by the choice of k, l and $v_{s-1} \notin N(v_{l-1})$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{s-1}, v_{l-1}, v_{l-2}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_l, x.$$

So let v_{s-1} correspond to v_s for the case $k+1 < s < l-1$.

Now let $l < s < t$. Again by the choice of t, $v_s \notin N(v_1)$. So assume $v_s \in N(v_t)$. Now $v_{s+1} \notin N(x)$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_s, v_t, v_{t-1}, \ldots, v_{s+1}, x.$$

Also $v_{s+1} \notin N(v_{l-1})$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{l-1}, v_{s+1}, v_{s+2}, \ldots, v_t, v_s, v_{s-1}, \ldots, v_l, x.$$

So for $l < s < t$ let v_{s+1} be the correspondent of v_s.

Finally suppose $t < s \leq m$.

Recall that $v_s \in N(v_1) \cup N(v_t)$. We will first suppose $v_s \in N(v_1)$, and show that $v_s \notin N(v_{l-1}) \cup N(x)$.

First suppose $s < m$. Then $v_s \notin N(x)$ for if it were we would have $v_{s-1}v_{s+1} \in E(G)$ since G is claw-free and get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{s+1}, v_{s-1}, v_{s-2}, \ldots, v_1, v_s, x.$$

Now suppose $s = m$. Then $v_s \notin N(x)$ for if it were we would have $v_1, v_{m-1}, x \in N(v_m)$. Now G is claw-free and $v_1x, v_{m-1}x \notin E(G)$ so we must have $v_{m-1} \in N(v_1)$. But then we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_1, v_{m-1}, v_{m-2}, \ldots, v_{k+1}.$$

So for $v_s \in N(v_1)$ for $t < s \leq m$ we have $v_s \notin N(x)$.

Recall that $t < s \leq m$ and $v_s \in N(v_1)$. We have $v_s \notin N(v_{l-1})$ for if it were we would have $v_1, v_{l-1}, v_{s-1} \in N(v_s)$. Since G is claw-free and $v_1v_{l-1} \notin E(G)$ we have either v_1v_{s-1} or $v_{l-1}v_{s-1} \in E(G)$. But if $v_1v_{s-1} \in E(G)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_{l-1}, v_{l-2}, \ldots, v_1, v_{s-1}, v_{s-2}, \ldots, v_l, x$$

and if $v_{l-1}v_{s-1} \in E(G)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_1, v_2, \ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_l, x.$$

So we have shown that for $v_s \in N(v_1)$, $t < s \leq m$ we have $v_s \notin N(v_{l-1}) \cup N(x)$.

Next suppose $v_s \in N(v_t)$ where $t < s \leq m$. Again, we will show that $v_s \notin N(v_{l-1}) \cup N(x)$. Now we can assume $s > t+1$, since if $s = t+1$ we have $v_s = v_{t+1} \in N(v_1)$, and we have just shown $v_s = v_{t+1} \notin N(v_{l-1}) \cup N(x)$.

First suppose $t+1 < s < m$. Then $v_s \notin N(x)$ for if it were we would have $v_{s-1}v_{s+1} \in E(G)$ since G is claw-free and get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{s+1}, v_{s-1}, v_{s-2}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, v_s, x.$$

Now suppose $s = m$. Then $v_s \notin N(x)$ for if it were we would have $v_t, v_{m-1}, x \in N(v_m)$. Now G is claw-free and $v_t x, v_{m-1} x \notin E(G)$ so we must have $v_{m-1} \in N(v_t)$. But then we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{m-1}, v_t, v_{t-1}, \ldots, v_{k+1}.$$

So for $v_s \in N(v_t)$ and $t+1 < s \leq m$, we have shown $v_s \notin N(x)$.

Again recall $t+1 < s \leq m$, where $v_s \in N(v_t)$. We have $v_s \notin N(v_{l-1})$ for if it were we would have $v_t, v_{l-1}, v_{s-1} \in N(v_s)$. Since G is claw-free and $v_t v_{l-1} \notin E(G)$ we have either $v_t v_{s-1}$ or $v_{l-1} v_{s-1} \in E(G)$. But if $v_t v_{s-1} \in E(G)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_{l-1}, v_{l-2}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{s-1}, v_t, v_{t-1}, \ldots, v_l, x$$

and if $v_{l-1} v_{s-1} \in E(G)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_{s-1}, v_{s-2},$$

$$\ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-1}.$$

For $v_s \in N(v_t)$, $t+1 < s \leq m$ we have that $v_s \notin N(v_{l-1})$. Thus for $t < s \leq m$ let v_s be the correspondent of v_s in our 1:1 mapping.

We claim we have found a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{l-1}) \cup N(x)) - \{v_1, v_{l-1}, x\}$. Clearly $v_1, v_{l-1}, x \notin N(v_{l-1}) \cup N(x)$. We have shown a 1:1 mapping from $N(v_1) \cup N(v_t)$ to a subset S of $V(G) - (N(v_{l-1}) \cup N(x))$. We now claim $v_1, v_{l-1}, x \notin S$.

Suppose $v_1 \in S$. Then $v_1 = v_s, v_{s-1}$ or v_{s+1} where $v_s \in N(v_1) \cup N(v_t)$. But $v_1 = v_s$ implies $s = 1$, $v_1 = v_{s-1}$ implies $s = 2 > k+1$ and $v_1 = v_{s+1}$ implies $s = 0$, all contradictions. Thus $v_1 \notin S$.

Next suppose $v_{l-1} \in S$. Then $v_{l-1} = v_s, v_{s-1}$ or v_{s+1} for some s. But $v_{l-1} = v_s$ implies $s = l-1$, $v_{l-1} = v_{s-1}$ implies $s = l$ and $v_{l-1} = v_{s+1}$ implies $s = l-2 > l$, all contradictions. Thus $v_{l-1} \in S$.

Finally $x \notin S$ for all the image points are in P and $x \notin P$.

We get

$$(p-3)/2 < |N(v_1) \cup N(v_t)|$$

$$\leq |V(G) - (N(v_{l-1}) \cup N(x)) - \{v_1, v_{l-1}, x\}|$$

$$< p - (p-3)/2 - 3 = (p-3)/2$$

a contradiction.

Case 1.2: Suppose $k+1 < t < l-1$. Again, we will arrive at a contradiction by showing that there is a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{l-1}) \cup N(x))$. Recall that if $y \in N(v_1) \cup N(v_t)$ then $y \in P$ and $y = v_s$ for some s. We have shown that $s \neq 1, t, k, l-1, l$. We now claim $s \neq l+1, l+2$. For suppose $v_{l+1} \in N(v_1)$, then we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{l+1}, v_1, v_2, \ldots, v_l, x$$

and if $v_{l+2} \in N(v_1)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{l+2}, v_1, v_2, \ldots, v_{l-1}, v_{l+1}, v_l, x.$$

(Recall that $v_{l-1}v_{l+1} \in E(G)$ since G is claw-free.) Also if $v_{l+1} \in N(v_t)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{l+1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_l, x$$

and if $v_{l+2} \in N(v_t)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{l+2}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_{l+1}, v_l, x.$$

Thus $s \neq l+1, l+2$.

We will now consider 2 cases:

Case 1.2.1: Assume v_{l-1} is adjacent to some vertex v_q where $q < k$.

As in Case 1.1.1, we have $q < k-2$.

We will now show $s \neq k-1, k-2$ where $v_s \in N(v_1) \cup N(v_t)$.

Suppose $v_{k-1} \in N(v_1)$. Then we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_q, v_{q-1}, \ldots, v_1, v_{k-1}, v_{k-2}, \ldots, v_{q+1}.$$

Now suppose $v_{k-2} \in N(v_1)$. Then we will get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_{l-1}, v_q, v_{q-1}, \ldots, v_1, v_{k-1}, v_{k-2}, \ldots, v_{q+1}.$$

Next suppose $v_{k-1} \in N(v_t)$. Then we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-1}, v_t, v_{t-1}, \ldots, v_k, x.$$

Finally suppose $v_{k-2} \in N(v_t)$. Then we will get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-2}, v_t, v_{t-1}, \ldots, v_{k+1}, v_{k-1}, v_k, x.$$

Thus $s \neq k-1, k-2$.

We will now construct a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{l-1}) \cup N(x))$. Recall that $v_s \in N(v_1) \cup N(v_t)$.

Suppose $s \leq r$. Then $v_s \in N(v_1)$. Now $v_{s-1} \notin N(x)$ by the choice of k and $v_{s-1} \notin N(v_{l-1})$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_s, v_1, v_2, \ldots, v_{s-1}, v_{l-1}, v_{l-2}, \ldots, v_{k+1}.$$

So for $s \leq r$, let v_{s-1} be the correspondent of v_s in our 1:1 mapping.

Now suppose $r < s < k-2$. Then $v_s \notin N(v_1)$ and therefore $v_s \in N(v_t)$ by the choice of r, t. By the choice of k, $v_{s+1} \notin N(x)$. Also $v_{s+1} \notin N(v_{l-1})$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_{s+1}, v_{l-1}, v_{l-2}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_s, v_t, v_{t-1}, \ldots, v_{k+1}.$$

Thus for $r < s < k-2$, let v_{s+1} correspond to v_s in the 1:1 mapping.

Next suppose $k < s < t$. Then by the choice of t, $v_s \notin N(v_1)$. Suppose $v_s \in N(v_t)$. Then $v_{s+1} \notin N(x)$ by the choice of l and $v_{s+1} \notin N(v_{l-1})$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_{s+1}, v_{s+2},$$

$$\ldots, v_t, v_s, v_{s-1}, \ldots, v_{k+1}.$$

So let v_{s+1} be the correspondent of v_s when $k < s < t$.

Finally suppose $t < s < l-1$ or $l+2 < s \leq m$.

First suppose $v_s \in N(v_1)$. Then $v_{s-1} \notin N(v_{l-1}) \cup N(x)$ as above. So suppose $v_s \in N(v_t)$. Then $v_{s-1} \notin N(x)$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{s-1}, x.$$

Also $v_{s-1} \notin N(v_{l-1})$ for suppose not. Then for $s < l-1$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{s-1}, v_{l-1}, v_{l-2},$$

$$\ldots, v_s, v_t, v_{t-1}, \ldots, v_{k+1}$$

and for $s > l+2$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_l, x.$$

So for $t < s < l-1$ or $l+2 < s \leq m$, let v_{s-1} correspond to v_s in the 1:1 mapping.

Note that v_t has been chosen as an image point twice, once for the case $k < s < t$ and again for the case $t < s < l-1$. We claim we have shown the existence of a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to $V(G) - (N(v_{l-1}) \cup N(x)) - \{v_{k-1}, v_{k+1}, v_{l-1}, x\}$. Clearly we have shown a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to a subset S of $V(G) - (N(v_{l-1}) \cup N(x))$. We now claim $v_{k-1}, v_{k+1}, v_{l-1}, x \notin S$.

Suppose $v_{k-1} \in S$. Then $v_{k-1} = v_{s-1}$ or v_{s+1} where $v_s \in N(v_1) \cup N(x)$. But $v_{k-1} = v_{s-1}$ implies $s = k$ and $v_{k-1} = v_{s+1}$ implies $s = k-2$, both contradictions. Thus $v_{k-1} \notin S$.

Next suppose $v_{k+1} \in S$. Then $v_{k+1} = v_{s-1}$ or v_{s+1} for some s. But $v_{k+1} = v_{s-1}$ implies $s = k+2 \leq r$ or $s = k+2 > t$, both contradictions since $r < k$ and $t > k+1$. Also $v_{k+1} = v_{s+1}$ implies $s = k$ another contradiction. Thus $v_{k+1} \notin S$.

Now suppose $v_{l-1} \in S$. Then $v_{l-1} = v_{s-1}$ or v_{s+1} for some s. But $v_{l-1} = v_{s-1}$ implies $s = l$ and $v_{l-1} = v_{s+1}$ implies $s = l-2 < t$, both contradictions. Thus $v_{l-1} \notin S$.

Finally $x \notin S$ since all the image points are in P and $x \notin P$.

Clearly $v_{l-1}, x \notin N(v_{l-1}) \cup N(x)$. Also $v_{k-1}, v_{k+1} \notin N(x)$ and earlier in this case we showed $v_{k-1} \notin N(v_{l-1})$. Now $v_{k+1} \notin N(v_{l-1})$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_{k+1}, v_{k+2}, \ldots, v_t.$$

Thus $v_{k-1}, v_{k+1}, v_{l-1}, x \notin N(v_{l-1}) \cup N(x)$.

Thus we have

$$(p-3)/2 - 1 < |N(v_1) \cup N(v_t) - v_{t-1}|$$

$$\leq |V(G) - (N(v_{l-1}) \cup N(x)) - \{v_{k-1}, v_{k+1}, v_{l-1}, x\}|$$

$$< p - (p-3)/2 - 4 = (p-3)/2 - 1$$

a contradiction.

Case 1.2.2: So we can suppose v_{l-1} is not adjacent to any vertex q where $q < k$.

Let $v_s \in N(v_1) \cup N(v_t)$. We will construct a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{l-1}) \cup N(x))$.

First suppose $1 < s < k$. Then by the choice of k, $v_s \notin N(x)$ and by the hypothesis $v_s \notin N(v_{l-1})$. So for $1 < s < k$, let v_s correspond to itself in the 1:1 mapping.

Next suppose $k < s < t$. Then by the choice of t, $v_s \notin N(v_1)$ so assume $v_s \in N(v_t)$. Now $v_{s+1} \notin N(x)$ by the choice of k, l and $v_{s+1} \notin N(v_{l-1})$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_{s+1}, v_{s+2},$$

$$\ldots, v_t, v_s, v_{s-1}, \ldots, v_{k+1}.$$

For $k < s < t$, let v_{s+1} correspond to v_s in our 1:1 mapping.

Finally suppose $t < s < l-1$ or $l+2 < s \leq m$.

First suppose $v_s \in N(v_1)$. Then as above, $v_{s-1} \notin N(v_{l-1}) \cup N(x)$. So suppose $v_s \in N(v_t)$. Then $v_{s-1} \notin N(x)$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{s-1}, x.$$

Also $v_{s-1} \notin N(v_{l-1})$ or else for $s < l-1$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{s-1}, v_{l-1}, v_{l-2},$$

$$\ldots, v_s, v_t, v_{t-1}, \ldots, v_{k+1}$$

and for $s > l+2$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_l, x.$$

So for $t < s < l-1$ or $l+2 < s \leq m$, let v_{s-1} correspond to v_s in the 1:1 mapping.

Note that v_t has been chosen as an image point twice, once for the case $k < s < t$ and again for the case $t < s < l-1$. We claim we have shown the existence of a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to $V(G) - (N(v_{l-1}) \cup N(x)) - \{v_1, v_{k+1}, v_{l-1}, x\}$. Clearly we have shown a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to a subset S of $V(G) - (N(v_{l-1}) \cup N(x))$. We now claim $v_1, v_{k+1}, v_{l-1}, x \notin S$.

Suppose $v_1 \in S$. Then $v_1 = v_s, v_{s-1}$ or v_{s+1} where $v_s \in N(v_1) \cup N(v_t)$. But $v_1 = v_s$ implies $s = 1$, $v_1 = v_{s-1}$ implies $s = 2 > t$ and $v_1 = v_{s+1}$ implies $s = 0$, all contradictions. Thus $v_1 \notin S$.

Next suppose $v_{k+1} \in S$. Then $v_{k+1} = v_s, v_{s-1}$ or v_{s+1} for some s. But $v_{k+1} = v_s$ implies $s = k+1 < k$, $v_{k+1} = v_{s-1}$ implies $s = k+2 > t$ and $v_{k+1} = v_{s+1}$ implies $s = k$, all contradictions. Thus $v_{k+1} \notin S$.

Now suppose $v_{l-1} \in S$. Then $v_{l-1} = v_s, v_{s-1}$ or v_{s+1} for some s. But $v_{l-1} = v_s$ implies $s = l-1$ and $v_{l-1} = v_{s-1}$ implies $s = l$ both contradictions. Now $v_{l-1} = v_{s+1}$ implies $s = l-2 < t$, but by Case 1.2 $t < l-1$, a contradiction. Thus $v_{l-1} \notin S$.

Finally $x \notin S$ since all the image points are in P and $x \notin P$.

Clearly $v_1, v_{l-1}, x \notin N(v_{l-1}) \cup N(x)$. Also $v_{k+1} \notin N(x)$ and $v_{k+1} \notin N(v_{l-1})$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_{k+1}, v_{k+2}, \ldots, v_t.$$

Thus $v_1, v_{k+1}, v_{l-1}, x \notin N(v_{l-1}) \cup N(x)$.

We get

$$\begin{aligned} (p-3)/2 - 1 &< |N(v_1) \cup N(v_t) - v_{t-1}| \\ &\leq |V(G) - (N(v_{l-1}) \cup N(x)) - \{v_1, v_{k+1}, v_{l-1}, x\}| \\ &< p - (p-3)/2 - 4 = (p-3)/2 - 1 \end{aligned}$$

a contradiction.

Case 1.3: Suppose $t < k$.

Let $v_s \in N(v_1) \cup N(v_t)$. We have already shown $s \neq 1, t, k, l-1, l$. We will now show $s \neq k+1, k+2$ or $l+1$.

Now $v_{k+1} \notin N(v_1)$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_1, v_2, \ldots, v_k, x$$

and $v_{k+1} \notin N(v_t)$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_k, x.$$

Also $v_{k+2} \notin N(v_1)$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+2}, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, v_k, x$$

and $v_{k+2} \notin N(v_t)$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+2}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{k-1}, v_{k+1}, v_k, x.$$

Finally, $v_{l+1} \notin N(v_1)$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{l+1}, v_1, v_2, \ldots, v_l, x$$

and $v_{l+1} \notin N(v_t)$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{l+1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_l, x.$$

Thus $s \neq k+1, k+2$ or $l+1$.

We will now consider 2 cases:

Case 1.3.1: Suppose v_{l-1} is adjacent to some vertex v_q where $q < k$.

Now $q < k-2$ as in Case 1.1.1. Let $v_s \in N(v_1) \cup N(v_t)$. Now $s \neq k-1$ for if $v_{k-1} \in N(v_1)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_q, v_{q-1}, \ldots, v_1, v_{k-1}, v_{k-2}, \ldots, v_{q+1}.$$

Also $v_{k-1} \notin N(v_t)$ for if $t > q$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_q, v_{q+1}, \ldots, v_t, v_{k-1}, v_{k-2}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{q-1}$$

and if $t < q$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_q, v_{q-1}, \ldots, v_{t+1}, v_1, v_2,$$

$$\ldots, v_t, v_{k-1}, v_{k-2}, \ldots, v_{q+1}.$$

Finally, if $t = q$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{k-1}.$$

Thus $s \neq k-1$.

We will now construct a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{l-1}) \cup N(x))$.

First suppose $1 < s \leq r$. Then $v_s \in N(v_1)$. By the choice of k, $v_{s-1} \notin N(x)$ and $v_{s-1} \notin N(v_{l-1})$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_1, v_s, v_{s+1}, \ldots, v_{k-1}.$$

So for $1 < s \leq r$, let v_{s-1} correspond to v_s in our 1:1 mapping.

Now suppose $r < s < t$. Then by the choice of t we have $v_s \notin N(v_1)$ so assume $v_s \in N(v_t)$. By the choice of k, $v_{s+1} \notin N(x)$ and $v_{s+1} \notin N(v_{l-1})$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_{s+1}, v_{s+1}, \ldots, v_t, v_s, v_{s-1},$$

$$\ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{k-1}.$$

For $r < s < t$, let v_{s+1} be the correspondent of v_s in our 1:1 mapping.

Next suppose $t < s < k-1$. By the choice of k we have $v_{s-1} \notin N(x)$. If $v_s \in N(v_1)$ we have already shown $v_{s-1} \notin N(v_{l-1}) \cup N(x)$. If $v_s \in N(v_t)$ we have $v_{s-1} \notin N(v_{l-1})$ else if $s > t+1$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2},$$

$$\ldots, v_{s-1}, v_{l-1}, v_{l-2}, \ldots, v_{k+1}$$

and if $s = t + 1$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{k-1}.$$

So in the case $t < s < k - 1$, let v_{s-1} correspond to v_s.

Now suppose $k+2 < s < l-1$. By the choice of l we have $v_{s-1} \notin N(x)$. If $v_s \in N(v_1)$ then $v_{s-1} \notin N(v_{l-1}) \cup N(x)$. and if $v_s \in N(v_t)$ then $v_{s-1} \notin N(v_{l-1})$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, v_s, v_{s+1},$$

$$\ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_{k+1}.$$

In the case $k + 2 < s < l - 1$ let v_{s-1} be the correspondent of v_s in the 1:1 mapping.

Finally suppose $l + 1 < s \leq m$. Suppose $v_s \in N(v_1)$. Then $v_{s-1} \notin N(v_{l-1}) \cup N(x)$. So suppose $v_s \in N(v_t)$. Then $v_{s-1} \notin N(x)$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{s-1}, x$$

and $v_{s-1} \notin N(v_{l-1})$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_l, x.$$

So for $l + 1 < s \leq m$ let v_{s-1} be the vertex corresponding to v_s in the 1:1 mapping.

Note that we have considered v_t as an image point twice, once for $r < s < t$ and again for $t < s < k - 1$. We claim we have found a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to $V(G) - (N(v_{l-1}) \cup N(x) - \{v_{k-2}, v_{k-1}, v_{l-1}, x\}$.

Clearly we have shown a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to a subset S of $V(G) - (N(v_{l-1}) \cup N(x))$. We now claim $v_{k-2}, v_{k-1}, v_{l-1}, x \notin S$.

Suppose $v_{k-2} \in S$. Then $v_{k-2} = v_{s-1}$ or v_{s+1} where $v_s \in N(v_1) \cup N(v_t)$. But $v_{k-2} = v_{s-1}$ implies $s = k - 1$, a contradiction. Also $v_{k-2} = v_{s+1}$ implies $s = k - 3 < t$, so $t = k - 2$ or $k - 1$. But $t = k - 2$ implies $v_{t+1} = v_{k-1} \in N(v_1)$ and $t = k - 1$ implies $v_{t+1} = v_k \in N(v_1)$, both contradictions. Thus $v_{k-2} \notin S$.

Now suppose $v_{k-1} \in S$. Then $v_{k-1} = v_{s-1}$ or v_{s+1} for some s. But $v_{k-1} = v_{s-1}$ implies $s = k$, a contradiction. Also $v_{k-1} = v_{s+1}$ implies $s = k - 2 < t$ so $t = k - 1$. But this implies $v_{t+1} = v_k \in N(v_1)$, a contradiction. Thus $v_{k-1} \notin S$.

Next suppose $v_{l-1} \in S$. Then $v_{l-1} = v_{s-1}$ or v_{s+1} for some s. But $v_{l-1} = v_{s-1}$ implies $s = l$, a contradiction. Also $v_{l-1} = v_{s+1}$ implies $s = l - 2 < t$,so $t = l - 1 > k$, again giving a contradiction. Thus $v_{l-1} \notin S$.

Finally, $x \notin S$ since all image points are on P and $x \notin P$.

Clearly, $v_{l-1}, x \notin N(v_{l-1}) \cup N(x)$ and by the choice of k, $v_{k-2}, v_{k-1} \notin N(x)$. We will now show $v_{k-2}, v_{k-1} \notin N(v_{l-1})$.

If $v_{k-2} \in N(v_{l-1})$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_{l-1}, v_{k-2}, v_{k-3}, \ldots, v_1$$

(recall $v_{k-1}v_{k+1} \in E(G)$ since G is claw-free) and if $v_{k-1} \in N(v_{l-1})$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k+1}, \ldots, v_{l-1}, v_{k-1}, v_{k-2}, \ldots, v_1.$$

Thus $v_{k-1}, v_{k-2} \notin N(v_{l-1})$ and $v_{k-2}, v_{k-1}, v_{l-1}, x \notin N(v_{l-1}) \cup N(x)$.

We get

$$\begin{aligned}(p-3)/2 - 1 &< |N(v_1) \cup N(v_t) - v_{t-1}| \\ &\leq |V(G) - (N(v_{l-1}) \cup N(x)) - \{v_{k-2}, v_{k-1}, v_{l-1}, x\}| \\ &< p - (p-3)/2 - 4 = (p-3)/2 - 1\end{aligned}$$

a contradiction.

Case 1.3.2: So we can assume v_{l-1} is not adjacent to any vertex v_q where $q < k$. Recall that $t < k$.

Suppose $1 < s < t$ or $t < s < k$. Then by the choice of k, $v_s \notin N(x)$ and by hypothesis $v_s \notin N(v_{l-1})$. So for $1 < s < t$ or $t < s < k$ let v_s be its own correspondent in the 1:1 mapping.

Now suppose $k + 2 < s < l - 1$. By the choice of l, $v_{s-1} \notin N(x)$. If $v_s \in N(v_1)$ then $v_{s-1} \notin N(v_{l-1})$ and if $v_s \in N(v_t)$, then $v_{s-1} \notin N(v_{l-1})$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_l, x, v_k, v_{k-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, v_s, v_{s+1},$$

$$\ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_{k+1}.$$

For $k + 1 < s < l - 1$ let v_{s-1} correspond to v_s in the 1:1 mapping.

Finally, suppose $l + 1 < s \leq m$. Suppose $v_s \in N(v_1)$. Then $v_{s-1} \notin N(v_{l-1}) \cup N(x)$. So suppose $v_s \in N(v_t)$. Then $v_{s-1} \notin N(x)$ for if it were we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{s-1}, x$$

and $v_{s-1} \notin N(v_{l-1})$ or else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{l-1}, v_{s-1}, v_{s-2}, \ldots, v_l, x.$$

So for $l + 1 < s \leq m$, let v_{s-1} correspond to v_s in the 1:1 mapping.

We claim we have found a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{l-1}) \cup N(x)) - \{v_1, v_t, v_{l-1}, x\}$. Clearly $v_1, v_t, v_{l-1}, x \notin N(v_{l-1}) \cup N(x)$. We have shown a 1:1 mapping from $N(v_1) \cup N(v_t)$ to a subset S of $V(G) - (N(v_{l-1}) \cup N(x))$. We now claim $v_1, v_t, v_{l-1}, x \notin S$.

Suppose $v_1 \in S$. Then $v_1 = v_{s-1}$ or v_s where $v_s \in N(v_1) \cup N(v_t)$. But $v_1 = v_s$ implies $s = 1$ and $v_1 = v_{s-1}$ implies $s = 2 > k + 2$, both contradictions. Thus $v_1 \notin S$.

Now suppose $v_t \in S$. Then $v_t = v_{s-1}$ or v_s for some s. But $v_t = v_{s-1}$ implies $s = t + 1 > k + 2$ and $v_t = v_s$ implies $s = t$, both contradictions. Thus $v_t \notin S$.

Next suppose $v_{l-1} \in S$. Then $v_{l-1} = v_{s-1}$ or v_s for some s. But $v_{l-1} = v_{s-1}$ implies $s = l$ and $v_{l-1} = v_s$ implies $s = l - 1$, both contradictions. Thus $v_{l-1} \notin S$.

Finally, $x \notin S$ since all the image points are on P and $x \notin P$.

We get

$$\begin{aligned}(p-3)/2 &< |N(v_1) \cup N(v_t)| \\ &\leq |V(G) - (N(v_{l-1}) \cup N(x)) - \{v_1, v_t, v_{l-1}, x\}| \\ &< p - (p-3)/2 - 4 = (p-3)/2 - 1\end{aligned}$$

a contradiction.

Case 2: Now suppose $l = m$. Then x is not adjacent to any v_s where $s \neq k, l$. We look at two cases for t, where, as in Case 1, v_t is the vertex with the lowest subscript so that v_1 is adjacent to v_{t+1} and not to v_t.

We claim $t \neq k, k+1, m-1, m$. Now $t = k$ implies v_1 is adjacent to v_{k+1} in which case we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_1, v_2, \ldots, v_k, x.$$

Also $t = k+1$ implies v_1 is adjacent to v_{k+2} and then we would get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+2}, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, v_k, x.$$

(Recall $v_{k-1}v_{k+1} \in E(G)$ since G is claw-free.) Now if $t = m-1$, we have v_1 adjacent to v_m. But then $v_1, v_{m-1}, x \in N(v_m)$. Since G is claw-free and neither v_1 nor v_{m-1} is adjacent to x we must have $v_1 v_{m-1} \in E(G)$. But then we will get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_1, v_{m-1}, v_{m-2}, \ldots, v_{k+1}.$$

Thus $t \neq m-1$. Finally, $t \neq m$ since v_1 is adjacent to v_{t+1}.

Let $v_s \in N(v_1) \cup N(v_t)$. Now if $1 < s \leq r$ we have $v_s \in N(v_1)$ by the definition of r. By the definition of k, we have $v_{s-1} \notin N(x)$. Also, $v_{s-1} \notin N(v_{m-1})$ for if it were we would get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_s, v_1, v_2, \ldots, v_{s-1}, v_{m-1}, v_{m-2}, \ldots, v_{k+1}.$$

Thus for $1 < s \leq r$ we have $v_{s-1} \notin N(v_1) \cup N(v_t)$.

Let $v_s \in N(v_1)$. Then for $k+1 < s < m-1$ we clearly have $v_{s-1} \notin N(x)$. Also, $v_{s-1} \notin N(v_{l-1})$ else we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_1, v_s, v_{s+1}, \ldots, v_{m-1}, v_{s-1}, v_{s-2}, \ldots, v_{k+1}.$$

Case 2.1: Suppose $t < k$. We will show that there is a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{m-1}) \cup N(x)$.

Let $v_s \in N(v_1) \cup N(v_t)$. Clearly $s \neq 1, t$. Now $s \neq k$ since then, on the one hand, if $v_k \in N(v_1)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_{k-1}, v_{k-2}, \ldots, v_1, v_k, x$$

and on the other hand, if $v_k \in N(v_t)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_{k-1}, v_{k-2}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, v_k, x.$$

Similarly, $s \neq k+1$ since if $v_{k+1} \in N(v_1)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_1, v_2, \ldots, v_k, x$$

and if $v_{k+1} \in N(v_t)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_k, x.$$

Next, $s \neq m-1$ since if $v_{m-1} \in N(v_1)$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_1, v_{m-1}, v_{m-2}, \ldots, v_{k+1}$$

and if $v_{m-1} \in N(v_t)$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, v_{m-1}, v_{m-2}, \ldots, v_{k+1}.$$

Finally, we claim $s \neq m$.

If $v_m \in N(v_1)$ we get $\{v_1, v_{m-1}, x\} \in N(v_m)$, but these three vertices are independent contradicting the fact that G is claw-free. If $v_m \in N(v_t)$ then we would have the three independent vertices v_t, v_{m-1}, x all in $N(v_m)$. Thus $s \neq 1, t, k, k+1, m-1$ or m.

We will now consider 2 subcases:

Case 2.1.1: Assume v_{m-1} is adjacent to a vertex v_q with $q < k$.

Now $q \neq k-1$ since if $v_{k-1} \in N(v_{m-1})$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_{k-1}, v_{k-2}, \ldots, v_1.$$

Also $q \neq k-2$ since if $v_{k-2} \in N(v_{m-1})$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_{m-1}, v_{k-2}, v_{k-3}, \ldots, v_1.$$

(Note $v_{k-1}v_{k+1} \in E(G)$ since G is claw-free.) Thus $q < k-2$.

Consider $v_s \in N(v_1) \cup N(v_t)$. We have already shown that $s \neq 1, t, k, k+1, m-1$ or m. We now claim $s \neq k-1$ or $k-2$.

Suppose $s = k-1$. If $v_{k-1} \in N(v_1)$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_q, v_{q-1}, \ldots, v_1, v_{k-1}, v_{k-2}, \ldots, v_{q+1}.$$

If $v_{k-1} \in N(v_t)$, for $q < t$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_q, v_{q+1}, \ldots, v_t, v_{k-1}, v_{k-2}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{q-1},$$

for $q > t$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_q, v_{q+1}, \ldots, v_{k-1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{q-1}$$

and for $q = t$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{k-1}.$$

Thus $s \neq k-1$.

Now suppose $s = k-2$. If $v_{k-2} \in N(v_1)$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, v_{k+1}, \ldots, v_{m-1}, v_q, v_{q-1}, \ldots, v_1, v_{k-2}, v_{k-3}, \ldots, v_{q+1}.$$

If $v_{k-2} \in N(v_t)$, for $q < t$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, v_{k+1}, \ldots, v_{m-1}, v_q, v_{q+1}, \ldots, v_t, v_{k-2}, v_{k-3}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{q-1}$$

and for $q > t$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, v_{k+1}, \ldots, v_{m-1}, v_q, v_{q+1}, \ldots, v_{k-2}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{q-1}.$$

If $q = t$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_{m-1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{k-2}.$$

Thus $s \neq k-2$.

We will now construct a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{m-1}) \cup N(x))$. Let $v_s \in N(v_1) \cup N(v_t)$.

First suppose $1 < s \leq r$. Then $v_{s-1} \notin N(v_{m-1}) \cup N(x)$ as above. So for $1 < s \leq r$, let v_{s-1} correspond to v_s in the 1:1 mapping.

Next suppose $r+1 \leq s < t$. Since $v_s \notin N(v_1)$ by the choice of t, we have $v_s \in N(v_t)$ and $v_{s+1} \notin N(x)$ by the choice of k. Also $v_{s+1} \notin N(v_{m-1})$ for of it were we would get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_{s+1}, v_{s+2}, \ldots, v_t, v_s, v_{s-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{k-1}.$$

So for the case $r+1 \leq s < t$, let v_{s+1} be the correspondent of v_s.

Now suppose $t < s < k-2$. Then $v_{s-1} \notin N(x)$ by the choice of k.

Suppose $v_s \in N(v_1)$. Then $v_{s-1} \notin N(v_{m-1})$ for if it were we would get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_{s-1}, v_{s-2}, \ldots, v_1, v_s, v_{s+1}, \ldots, v_{k-1}.$$

Now suppose $v_s \in N(v_t)$. Then $v_{s-1} \notin N(v_{m-1})$ or else we for $s > t+1$ get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_{s-1}, v_{s-2}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, v_s, v_{s+1}, \ldots, v_{k-1}$$

and for $s = t + 1$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{k-1}.$$

So let v_{s-1} be the correspondent of v_s when $t < s < k - 2$.

Finally suppose $k + 1 < s < m - 1$. Then $v_{s-1} \notin N(x)$ by the definition of l.

Then if $v_s \in N(v_1)$ we have already shown $v_{s-1} \notin N(v_{m-1})$. Also, if $v_s \in N(v_t)$ we have $v_{s-1} \notin N(v_{m-1})$ for if it were we would get the longer v_m-path

$$v_m, x, v_k, u_{k+1}, \ldots, v_{s-1}, v_{m-1}, v_{m-2}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{k-1}.$$

So for $k + 2 < s < m - 1$, let v_{s-1} be the vertex corresponding to v_s in the 1:1 mapping.

Note that v_t has been used as an image point twice, once for the case $r + 1 \leq s < t$ and again where $t < s < k - 2$. We claim to have found a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to $V(G) - (N(v_{m-1}) \cup N(x)) - \{v_{k-2}, v_{k-1}, v_{m-1}, x\}$. Now we have shown that $v_{k-2}, v_{k-1}, v_{m-1}, x \notin N(v_{m-1}) \cup N(x)$. (Recall that at the beginning of Case 2.1.1 we showed that $v_{k-2}, v_{k-1} \notin N(v_{m-1})$. Clearly we have shown a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to a subset S of $V(G) - (N(v_{m-1}) \cup N(x))$ we now claim $v_{k-2}, v_{k-1}, v_{m-1}, x \notin S$.

Suppose $v_{k-2} \in S$. Then $v_{k-2} = v_{s-1}$ or v_{s+1} where $v_s \in N(v_1) \cup N(v_t)$. But $v_{k-2} = v_{s-1}$ implies $s = k - 1$ a contradiction. Also $v_{k-2} = v_{s+1}$ implies $s = k - 3 < t$, so $t = k - 2$ or $k - 1$. But $t = k - 2$ implies v_1 is adjacent to v_{k-1} and $t = k - 1$ implies v_1 is adjacent to v_k, both false. Thus $v_{k-2} \notin S$.

Next suppose $v_{k-1} \in S$. Then $v_{k-1} = v_{s-1}$ or v_{s+1} where $v_s \in N(v_1) \cup N(v_t)$. But $v_{k-1} = v_{s-1}$ implies $s = k$, a contradiction. Also $v_{k-1} = v_{s+1}$ implies $s = k - 2 < t$, so $t = k - 1$. But $t = k - 1$ implies v_1 is adjacent to v_k, a contradiction. Thus $v_{k-1} \notin S$.

Now suppose $v_{m-1} \in S$. Then $v_{m-1} = v_{s-1}$ or v_{s+1} for some s with $v_s \in N(v_1) \cup N(v_t)$. But $v_{m-1} = v_{s-1}$ implies $s = m$ and $v_{m-1} = v_{s+1}$ implies $s = m - 2 < t$, both contradictions. Thus $v_{m-1} \notin S$.

Finally suppose $x \in S$. Then x is the image point of some $v_s \in N(v_1) \cup N(v_t)$. But all the image points are on P and $x \notin P$. Thus $x \notin S$.

We get

$$(p-3)/2 - 1 < |N(v_1) \cup N(v_t) - v_{t-1}|$$

$$\leq |V(G) - (N(v_{m-1}) \cup N(x)) - \{v_{k-2}, v_{k-1}, v_{m-1}, x\}|$$

$$< p - (p-3)/2 - 4 = (p-3)/2 - 1$$

a contradiction.

Case 2.1.2: So we can assume v_{m-1} is not adjacent to any vertex v_q with $q < k$.

First suppose $s \leq k - 1$. Then $v_s \notin N(x)$ by the choice of k and by hypothesis $v_s \notin N(v_{m-1})$. Thus for $s \leq k - 1$, let v_s be the vertex corresponding to v_s in the 1:1 mapping.

Next suppose $k + 1 < s < m - 1$. Then $v_{s-1} \notin N(x)$ by the choice of l.

Suppose $v_s \in N(v_1)$. Then $v_{s-1} \notin N(v_{m-1})$ as above. So suppose $v_s \in N(v_t)$. Then $v_{s-1} \notin N(v_{m-1})$ for if it were we would get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_t, v_s, v_{s+1}, \ldots, v_{m-1}, v_{s-1}, v_{s-2}, \ldots, v_{k+1}.$$

So for $k+1 < s < m-1$ let v_{s-1} correspond to v_s in the 1:1 mapping.

We claim we have shown the existence of a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{m-1}) \cup N(x)) - \{v_1, v_t, v_{m-1}, x\}$. Note that $v_1, v_t, v_{m-1}, x \notin N(v_{m-1}) \cup N(x)$. Clearly we have shown a 1:1 mapping from $N(v_1) \cup N(v_t)$ to a subset S of $V(G) - (N(v_{m-1}) \cup N(x))$. We now claim $v_1, v_t, v_{m-1}, x \notin S$.

First suppose $v_1 \in S$. Then $v_1 = v_s$ or v_{s-1} where $v_s \in N(v_1) \cup N(v_t)$. But $v_1 = v_s$ implies $s = 1$ and $v_1 = v_{s-1}$ implies $s = 2 > k+1$, both contradictions. So $v_1 \notin S$.

Next suppose $v_t \in S$. Then $v_t = v_s$ or v_{s-1} for some $v_s \in N(v_1) \cup N(v_t)$. But if $v_s = v_t$ we get $s = t$, a contradiction and if $v_t = v_{s-1}$ we get $s = t+1 > k+1$, so $t > k$, again a contradiction. Thus $v_t \notin S$.

Now suppose $v_{m-1} \in S$. Then $v_{m-1} = v_s$ or v_{s-1} for some s. But $v_{m-1} = v_s$ implies $s = m-1$ and $v_{m-1} = v_{s-1}$ implies $s = m$, both contradictions. Thus $v_{m-1} \notin S$.

Finally suppose $x \in S$. Then $x = v_s$ or v_{s-1}, but $x \notin P$. So $x \notin S$.

We get the following:

$$\begin{aligned} (p-3)/2 &< |N(v_1) \cup N(v_t)| \\ &\leq |V(G) - (N(v_{m-1}) \cup N(x)) - \{v_1, v_t, v_{m-1}, x\}| \\ &< p - (p-3)/2 - 4 = (p-3)/2 - 1 \end{aligned}$$

a contradiction.

Case 2.2: Suppose $k+1 < t < m-1$.

We will show that there is a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{m-1}) \cup N(x))$.

Recall $v_s \in N(v_1) \cup N(v_t)$. Clearly $s \neq 1$ or t.

We claim $s \neq k, m-1$ or m.

We first claim $s \neq k$. If $v_k \in N(v_1)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{k+1}, v_{k-1}, v_{k-2}, \ldots, v_1, v_k, x$$

and if $v_k \in N(v_t)$ we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_t, v_k, x.$$

(Recall that $v_{k-1}v_{k+1} \in E(G)$ since G is claw-free.) Thus $s \neq k$.

Next, we claim $s \neq m-1$. If $v_{m-1} \in N(v_1)$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_1, v_2, \ldots, v_{k-1}$$

and if $v_{m-1} \in N(v_t)$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_t, v_{m-1}, v_{m-2}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-1}.$$

Thus $s \neq m-1$.

Finally we claim $s \neq m$. If $v_m \in N(v_1)$, we get $v_1, v_{m-1}, x \in N(v_m)$. But these three vertices are independent, a contradiction to the fact that G is claw-free. Also if $v_m \in N(v_t)$, we get $v_t, v_{m-1}, x \in N(v_m)$ and again these are independent contradicting the fact that G is claw-free. Thus $s \neq m$.

Let $v_s \in N(v_1) \cup N(v_t)$ with $t < s < m-1$. We have already shown that if $v_s \in N(v_1)$, then for $k+1 < s < m-1$ we have $v_{s-1} \notin N(v_{m-1}) \cup N(x)$. In particular, if $s = t+1$ then $v_{s-1} \notin N(v_{m-1} \cup N(x)$. So suppose $v_s in N(v_t)$ with $t+1 < s < m-1$. Then by the choice of l, $v_{s-1} \notin N(x)$ and $v_{s-1} \notin N(v_{m-1}$ for if it were we would get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{s-1}, v_{m-1}, v_{m-2}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_{k+1}.$$

Thus, if $v_s \in N(v_1) \cup N(v_t)$ with $t < s < m-1$ we have $v_{s-1} \notin N(v_{m-1}) \cup N(x)$.

We now consider two cases:

Case 2.2.1: Assume v_{m-1} is adjacent to some vertex v_q where $q < k$. Then as in Case 2.1.1, $q \neq k-2, k-1$.

Suppose $v_s \in N(v_1) \cup N(v_t)$. We have already shown that $s \neq 1, t, k, m-1$ or m. We now claim $s \neq k-1$ or $k-2$. Now $v_{k-1} \notin N(v_1)$ for if it were we would get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_{m-1}, v_q, v_{q-1}, \ldots, v_1, v_{k-1}, v_{k-2}, \ldots, v_{q+1}$$

and $v_{k-1} \notin N(v_t)$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-1}, v_t, v_{t-1}, \ldots, v_k, x.$$

Thus $s \neq k-1$.

Next $v_{k-2} \notin N(v_1)$ for if it were we would get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, v_{k+1}, v_{k+2}, \ldots, v_{m-1}, v_q, v_{q-1}, \ldots, v_1, v_{k-2}, v_{k-3}, \ldots, v_{q+1}$$

and $v_{k-2} \notin N(v_t)$ else we get the longer v_m-path

$$v_m, v_{m-1}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-2}, v_t, v_{t-1}, \ldots, v_{k+1}, v_{k-1}, v_k, x.$$

Thus $s \neq k-2$.

We will now construct a 1:1 mapping from $N(v_1) \cup N(v_t)$ to $V(G) - (N(v_{m-1}) \cup N(x))$. Recall that $v_s \in N(v_1) \cup N(v_t)$.

First suppose $1 < s \leq r$. Then as above $v_{s-1} \notin N(v_{m-1}) \cup N(x)$. So for $1 < s \leq r$, let v_{s-1} correspond to v_s in the 1:1 mapping.

Now suppose $r+1 \leq s < k-2$ or $k < s < t$. Then by the choice of t and the definition of r, $v_s \notin N(v_1)$ so suppose $v_s \in N(v_t)$. Now $v_{s+1} \notin N(x)$ by the choice of k. Also $v_{s+1} \notin N(v_{m-1})$ else for $s < k-2$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_t, v_s, v_{s-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{m-1}, v_{s+1}, v_{s+2}, \ldots, v_{k-1}$$

and for $s > k$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{m-1}, v_{s+1}, v_{s+2}, \ldots, v_t, v_s, v_{s-1}, \ldots, v_{k+1}.$$

So if either $r+1 \leq s < k-2$ or $k < s < t$, let v_{s+1} be the correspondent of v_s.

Finally suppose $t < s < m-1$. Then as above, we have $v_{s-1} \notin N(v_{m-1}) \cup N(x)$. So for $t < s < m-1$, let v_{s-1} be the vertex corresponding to v_s in the 1:1 mapping.

Note that v_t has been chosen as an image point twice, once for the case $k < s < t$ and again for $t < s < m-1$. We claim we have found a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to $V(G) - (N(v_{m-1}) \cup N(x)) - \{v_{k-1}, v_{k+1}, v_{m-1}, x\}$. Clearly we have shown a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to a subset S of $V(G) - (N(v_{m-1}) \cup N(x))$. We now claim $v_{k-1}, v_{k+1}, v_{m-1}, x \notin S$.

First suppose $v_{k-1} \in S$. Then $v_{k-1} = v_{s-1}$ or v_{s+1} where $v_s \in N(v_1) \cup N(v_t)$. But $v_{k-1} = v_{s-1}$ implies $s = k$ and $v_{k-1} = v_{s+1}$ implies $s = k-2$, both contradictions. Thus $v_{k-1} \notin S$.

Next suppose $v_{k+1} \in S$. Then $v_{k+1} = v_{s-1}$ or v_{s+1} for some s. But $v_{k+1} = v_{s-1}$ implies $s = k+2 \leq r$ contradicting the fact that $r < k$, or $s = k+2 > t$ contradicting the hypothesis of Case 2.2 that $t > k+1$. Also $v_{k+1} = v_{s+1}$ implies $s = k$, a contradiction. Thus $v_{k+1} \notin S$.

Now suppose $v_{m-1} \in S$. Then $v_{m-1} = v_{s-1}$ or v_{s+1} for some s with $v_s \in N(v_1) \cup N(v_t)$. But $v_{m-1} = v_{s-1}$ implies $s = m$ a contradiction. Also $v_{m-1} = v_{s+1}$ implies $s = m-2 < t$, but $t < m-1$ by the hypothesis of Case 2.2. another contradiction. Thus $v_{m-1} \notin S$.

Finally $x \notin S$ since all the image points are on P and $x \notin P$.

Now clearly $v_{m-1}, x \notin N(v_{m-1}) \cup N(x)$ and $v_{k-1}, v_{k+1} \notin N(x)$. We will now show that $v_{k-1}, v_{k+1} \notin N(v_{m-1})$. If $v_{k-1} \in N(v_{m-1})$ we get the longer v_m-path

$$v_m, x, v_k, v_{k+1} \ldots, v_{m-1}, v_{k-1}, v_{k-2}, \ldots, v_1.$$

Now if $v_{k+1} \in N(v_{m-1})$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2} \ldots, v_{m-1}, v_{k+1}, v_{k+2}, \ldots, v_t.$$

Thus $v_{k-1}, v_{k+1}, v_{m-1}, x \notin N(v_{m-1}) \cup N(x)$.

We get

$$\begin{aligned}(p-3)/2 - 1 &< |N(v_1) \cup N(v_t) - v_{t-1}| \\ &\leq |V(G) - (N(v_{m-1}) \cup N(x)) - \{v_{k-1}, v_{k+1}, v_{m-1}, x\}| \\ &< p - (p-3)/2 - 4 = (p-3)/2 - 1\end{aligned}$$

a contradiction.

Case 2.2.2: So we can assume v_{m-1} is not adjacent to any vertex v_q with $q < k$.

Let $v_s \in N(v_1) \cup N(v_t)$. We will now construct our 1:1 mapping.

First suppose $s \leq k-1$. Then by the choice of k, $v_s \notin N(x)$ and by hypothesis $v_s \notin N(v_{m-1})$. So for $s \leq k-1$, let v_s be its own correspondent in the 1:1 mapping.

Next suppose $k < s < t$. Then by the choice of t, $v_s \notin N(v_1)$ so we will suppose $v_s \in N(v_t)$. By the choice of l we have $v_{s+1} \notin N(x)$. Also $v_{s+1} \notin N(v_{m-1})$ for if it were we would get the longer v_m-path

$$v_m, x, v_k, v_{k+1}, \ldots, v_s, v_t, v_{t-1}, \ldots, v_{s+1}, v_{m-1}, v_{m-2}, \ldots, v_{t+1}, v_1, v_2, \ldots, v_{k-1}.$$

For the case $k < s < t$, let v_{s+1} correspond to v_s in the 1:1 mapping.

Finally suppose $t < s < m-1$. Then $v_{s-1} \notin N(v_{m-1}) \cup N(x)$ as we have shown above. So for $t < s < m-1$, let v_{s-1} be the correspondent of v_s in the desired 1:1 mapping.

Note that v_t has been chosen as an image point twice, once for the case $k < s < t$ and again for $t < s < m-1$. We claim we have found a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to $V(G) - (N(v_{m-1}) \cup N(x)) - \{v_1, v_{k+1}, v_{m-1}, x\}$.

Clearly we have shown a 1:1 mapping from $N(v_1) \cup N(v_t) - v_{t-1}$ to a subset S of $V(G) - (N(v_{m-1}) \cup N(x))$. We now claim $v_1, v_{k+1}, v_{m-1}, x \notin S$.

First suppose $v_1 \in S$. Then $v_1 = v_s, v_{s-1}$ or v_{s+1} where $v_s \in N(v_1) \cup N(v_t)$. But $v_1 = v_s$ implies $s = 1$, $v_1 = v_{s+1}$ implies $s = 0$ and $v_1 = v_{s-1}$ implies $s = 2 > t$, all contradictions. Thus $v_1 \notin S$.

Now suppose $v_{k+1} \in S$. Then $v_{k+1} = v_s, v_{s-1}$ or v_{s+1} for some s with $v_s \in N(v_1) \cup N(v_t)$. But $v_{k+1} = v_s$ implies $s = k+1 \leq k-1$, a contradiction. Next $v_{k+1} = v_{s-1}$ implies $s = k+2 > t$, but $t > k+1$ by the hypothesis of Case 2.2. Finally $v_{k+1} = v_{s+1}$ implies $s = k$, a contradiction. Thus $v_{k+1} \notin S$.

Next suppose $v_{m-1} \in S$. Then $v_{m-1} = v_s, v_{s-1}$ or v_{s+1} for some s. But $v_{m-1} = v_s$ implies $s = m-1$ and $v_{m-1} = v_{s-1}$ implies $s = m$, both contradictions. Also $v_{m-1} = v_{s+1}$ implies $s = m-2 < t$, but $t < m-1$ by the hypothesis of Case 2.2. Thus $v_{m-1} \notin S$.

Finally, suppose $x \in S$. Then x is the image point of some $v_s \in N(v_1) \cup N(v_t)$. But all the image points are on P and $x \notin P$. Thus $x \notin S$.

Now clearly $v_1, v_{m-1}, x \notin N(v_{m-1}) \cup N(x)$ and $v_{k+1} \notin N(x)$. It remains to show that $v_{k+1} \notin N(v_{m-1})$. Now if $v_{k+1} \in N(v_{m-1})$ we get the longer v_m-path

$$v_m, x, v_k, v_{k-1}, \ldots, v_1, v_{t+1}, v_{t+2}, \ldots, v_{m-1}, v_{k+1}, v_{k+2}, \ldots, v_t.$$

Thus $v_1, v_{k+1}, v_{m-1}, x \notin N(v_{m-1}) \cup N(x)$.

We get

$$\begin{aligned} (p-3)/2 - 1 &< |N(v_1) \cup N(v_t) - v_{t-1}| \\ &\leq |V(G) - (N(v_{m-1}) \cup N(x)) - \{v_1, v_{k+1}, v_{m-1}, x\}| \\ &< p - (p-3)/2 - 4 = (p-3)/2 - 1 \end{aligned}$$

a contradiction. □

The graph in Figure 1 is 2-connected, claw-free and not homogeneously traceable. Here, $|N(u) \cup N(v)| = 2n + 2 = (p-4)/2$, so the bound in Theorem 6 is almost best possible.

The graph shown in Figure 2 is homogeneously traceable, with $|N(u) \cup N(v)| = (p-2)/2$, so Theorem 3 tells us nothing about this graph, whereas Theorem 6 tells us that it is homogeneously traceable.

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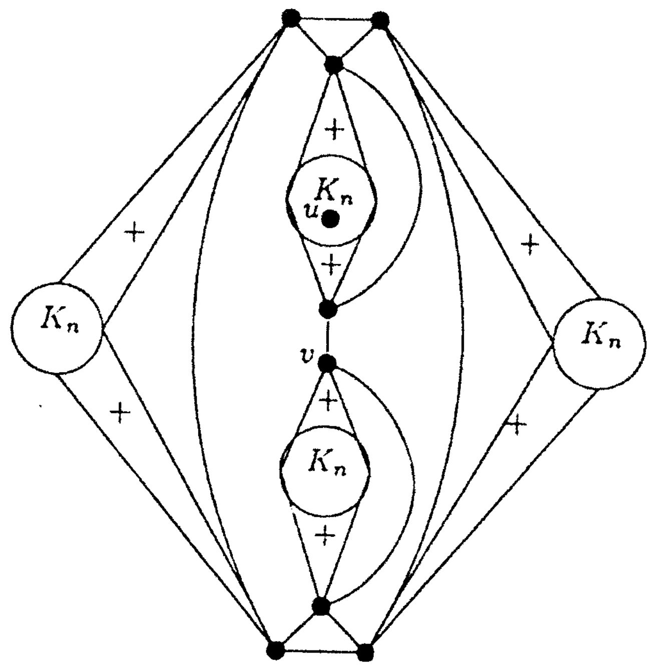

Figure 1

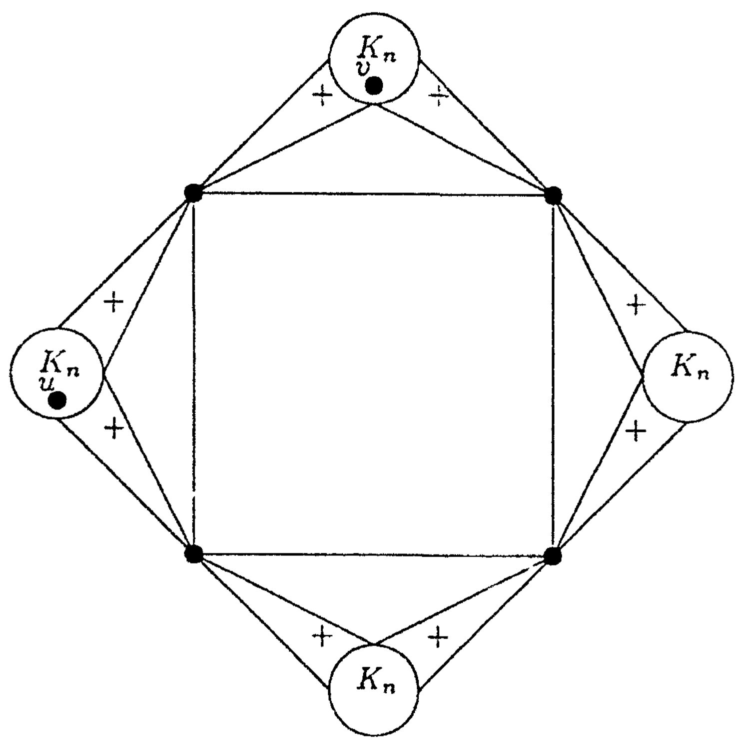

Figure 2